Bipartite Correlations through Enriched System Information in Phase-Coherent Amplitudes.

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Abstract:

This work presents a geometrically explicit representation of spin and polarization measurements in which the state of an individual particle is described by a continuous directional amplitude defined on the unit sphere. Measurement outcomes are modeled as local projection operations acting on these amplitudes, with statistical predictions obtained through averaging over preparation-dependent distributions.

Within this formulation, the correlation structure of entangled photon pairs is derived directly from phase-coherent amplitude overlaps, where phase information is encoded geometrically through relative orientation. The resulting expression reproduces the standard quantum correlation function $E(\alpha, \beta) = \cos\left(2(\alpha - \beta)\right)$ and exactly saturates the Tsirelson bound, without requiring additional assumptions beyond local projection geometry and phase tracking.

The analysis provides a unified geometric description of Stern–Gerlach filtering, polarization, and bipartite correlations, highlighting how phase coherence at the amplitude level governs the formation of quantum statistics prior to probability assignment.

1. Introduction

The quantum mechanical description of spin is traditionally formulated using discrete algebraic labels and operator algebra applied to ensembles. While this formalism is empirically successful, it largely suppresses the explicit geometric structure of individual particle amplitudes between preparation and measurement. In particular, the phase information that governs interference and correlation phenomena is typically encoded abstractly, rather than represented directly in terms of spatial geometry. In this work, we develop a formulation in which this phase structure is tracked explicitly through directional amplitudes on the unit sphere.

We explore the possibility that the spin state of an individual particle can be described more completely by a phase-coherent directional amplitude. In this refined picture, each particle carries a definite directional amplitude with an associated internal phase, consistent with the rotational symmetry of SU(2). To ground this definition physically, we re-analyze the Stern–Gerlach experiment [1], identifying the specific post-selection distribution required to reproduce the geometric transition probabilities discussed by

Feynman [2]. We find that an expanded definition of spin—one that treats amplitude density as a signed, phase-coherent distribution—is necessary to capture the full rotational structure of the single particle.

In this framework, measurement outcomes are not merely selected from a pre-existing set of eigenvalues but are generated by projections of this internal amplitude structure onto the analyzer settings. Statistical predictions then emerge from distributions over these phase-resolved amplitudes.

To rigorously validate whether this enriched kinematic description provides a faithful representation of quantum phenomenology, we apply it to the benchmark problem of entangled photon pairs. We ask a specific structural question: Can a strictly geometric framework, based on these enriched spin variables, reproduce the theoretical maximum for quantum correlations?

We show that this is indeed the case. For entangled photon pairs prepared with a shared geometric phase (orientation) at emission, we model the measurement process as an analyzer projection of the particle's internal amplitude. Correlations arise when the bilinear product of the responses is averaged over the shared preparation distribution. The resulting correlation function recovers the standard quantum cosine form and exactly saturates the Tsirelson bound [3].

This result suggests that the limitations typically associated with classical probabilistic models stem from the assumption that probabilities must factorize directly. By retaining phase coherence at the amplitude level prior to probability assignment, we recover the full quantum statistics within a geometrically consistent framework. This phase-resolved description offers a natural platform for understanding filtering, polarization, and interference within a unified kinematic setting.

2. Stern–Gerlach Revisited: Angular Amplitude Structure and Post-Selection

The Stern–Gerlach experiment [1] is commonly presented as a measurement revealing a pre-existing binary spin property. Operationally, however, the device functions as a state-preparation filter rather than a revelation of intrinsic values, as emphasized in both early and modern analyses: an initially unpolarized ensemble (beam) is spatially separated, and post-selection of only one output channel prepares a new spin state of the particles in this ensemble. Closely related geometric interpretations of spin preparation and rotation can be found in the analysis of spin and rotations presented in The Feynman Lectures [2], where the Stern–Gerlach experiment with its binary outcomes for spin-½ particles is treated as a paradigmatic example of quantum state preparation rather than classical orientation. In this section we show that this preparation can be represented by a **directional amplitude structure on the unit**

sphere, whose properties reproduce the standard quantum transition probabilities under rotation, while retaining more specific information.

2.1 Isotropic ensemble and analyzer geometry

Consider an initially isotropic ensemble of spin-½ particles incident on a Stern–Gerlach analyzer oriented along the z-axis. Let θ denote the polar angle between an arbitrary unit vector $\hat{v}$ on the Bloch sphere and the analyzer axis $\hat{z}$.

Post-selection of the "spin-up" output defines a prepared state. We associate to this state a **directional amplitude density** distribution of individual particles on the unit sphere S^2,

$$\rho(\theta) = \frac{1}{4\pi} + \frac{1}{2\pi}\cos\,\theta. \tag{2.1}$$

This object satisfies normalization requirement,

$$\int_{S^2} \rho(\theta)\, d\Omega = 1, \tag{2.2}$$

but it is not everywhere non-negative. In particular, $\rho(\theta) < 0$ for $\cos\,\theta < -1/2$.
It is important to emphasize that this specific signed density is not an arbitrary modeling choice. As shown in Appendix A, it is uniquely determined by rotational symmetry, normalization, and the operational requirement of hemisphere integration. No alternative angular dependence is compatible with these constraints. The appearance of negative values is essential for reproducing the correct rotational behavior and is therefore a necessary feature of the amplitude description rather than a defect, and will be discussed further below.

2.2 Interpretation of the density

The function $\rho(\theta)$ is **not** interpreted as a classical probability density over hidden spin directions. Instead:

1. ρ is a **phase-weighted directional amplitude density**, encoding the biased preparation imposed by the analyzer.
2. Negative values are admissible and reflect **destructive interference** of amplitudes associated with different orientations.
3. Observable probabilities arise only **after integration over analyzer-defined regions**, not by pointwise evaluation of ρ.

Accordingly, ρ plays a role analogous to a quasi-distribution: it is a bookkeeping / tracking device for amplitudes prior to projection.

2.3 Second analyzer and hemisphere integration

Let a second Stern–Gerlach analyzer be oriented along a unit vector $\hat{n}$ making an angle α with the z-axis. The analyzer accepts those components whose projection onto $\hat{n}$ is positive. Geometrically, this corresponds to the hemisphere

$$S_{\hat{n}} = \{\hat{v} \in S^2 : \hat{v} \cdot \hat{n} \geq 0\}. \tag{2.3}$$

The transition probability for a particle prepared in the z-up state to be transmitted by the second analyzer is given by the integral of the amplitude density over this hemisphere:

$$P(\hat{z} \uparrow \rightarrow \hat{n} \uparrow) = \int_{S_{\hat{n}}} \rho(\theta)\, d\Omega. \tag{2.4}$$

2.4 Explicit evaluation

Choose coordinates such that $\hat{n}$ lies in the xz-plane at polar angle α. Writing $\hat{v} \cdot \hat{n} = \cos\ \gamma$, the hemisphere condition is $\gamma \leq \pi/2$.

The integral (2.4) separates into two terms:

$$\int_{S_{\hat{n}}} \rho\, d\Omega = \frac{1}{4\pi} \int_{S_{\hat{n}}} d\Omega + \frac{1}{2\pi} \int_{S_{\hat{n}}} \cos\theta\, d\Omega = \frac{1}{2} + \frac{1}{2} \cos\ \alpha \tag{2.5}$$

a standard result obtained by expressing $\cos\ \theta = \hat{v} \cdot \hat{z}$ and exploiting rotational symmetry of the hemisphere. Substituting into (2.4),

$$P(\hat{z} \uparrow \rightarrow \hat{n} \uparrow) = \frac{1}{2} + \frac{1}{2} \cos\ \alpha = \cos^2\left(\frac{\alpha}{2}\right). \tag{2.6}$$

This is precisely the quantum-mechanical transition probability for sequential Stern–Gerlach measurements on a spin-½ system. Further details in Appendix A.

2.5 Role of negative values

The appearance of negative values in $\rho(\theta)$ is not a defect but a necessary feature. If ρ were constrained to be non-negative everywhere, the hemisphere integral (2.4) could not reproduce the $\cos^2(\alpha/2)$ law for all α. The negative regions of $\rho(\theta)$ encode a directional amplitude distribution tending towards giving an opposite direction. This ensures correct rotational behavior.

Thus, while individual directional amplitude contributions may carry negative weight, observable probabilities remain positive and properly normalized for a given spin direction. This structure mirrors the role of complex amplitudes in standard quantum mechanics, where **amplitude interference precedes probability assignment**. The appearance of negative regions is consistent with the use of quasi-distributions in quantum theory, and does not imply the existence of hidden classical variables.

2.6 Summary of the Stern–Gerlach result

We have shown that:

1. Post-selection by a Stern–Gerlach analyzer induces a directional amplitude structure on the Bloch sphere.
2. This structure is naturally represented by the signed density (2.1).
3. Integration over analyzer-defined hemispheres yields the correct quantum transition probabilities.

In the next section, the same projection logic governs photon polarization, leading to Malus' law [4].

3. Photon Polarization: Planar Amplitude Structure and Analyzer Projection

Photon polarization provides a natural extension of the Stern–Gerlach analysis to a two-dimensional setting. In contrast to spin-½ systems, polarization is confined to the transverse plane and is described by an axis rather than an oriented vector. These features lead to the characteristic double-angle dependence observed in polarization correlations. In this section we formulate polarization filtering as a planar amplitude structure acting on an analyzer projection and derive Malus' law in a form suitable for later analysis of entangled photon pairs.

3.1 Geometry of linear polarization

Consider a monochromatic photon propagating along the z-axis. Its polarization lies in the transverse (x, y)-plane. A linear polarization analyzer oriented at angle α defines a transmission axis

$$\hat{e}(\alpha) = \cos\ \alpha\, \hat{x} + \sin\ \alpha\, \hat{y}. \tag{3.1}$$

Because linear polarization corresponds to an axis, the directions $\hat{e}(\alpha)$ and $-\hat{e}(\alpha)$ represent the same physical state. Consequently, polarization observables are π-periodic in α.

3.2 Polarization as an amplitude overlap

A photon prepared in a linear polarization state aligned with α_0 is represented locally by the unit vector $\hat{e}(\alpha_0)$ in the transverse plane. A polarization analyzer at angle α acts as a linear projector onto $\hat{e}(\alpha)$. The transmission amplitude is therefore given by the scalar overlap

$$\mathcal{A}(\alpha_0 \to \alpha) = \hat{e}(\alpha) \cdot \hat{e}(\alpha_0) = \cos(\alpha - \alpha_0). \tag{3.2}$$

The corresponding transmission probability is the squared magnitude of this amplitude,

$$P(\alpha_0 \to \alpha) = |\, \mathcal{A} \,|^2 = \cos^2(\alpha - \alpha_0). \tag{3.3}$$

Equation (3.3) is Malus' law which acquires its quantum-mechanical interpretation through the Born rule [5], which identifies probabilities with squared amplitudes.

This result is obtained entirely locally: the analyzer modifies only the amplitude of the incoming field at its location, and the probability arises from the standard amplitude-to-intensity rule.

3.3 Phase tracking and rotation in the polarization plane

To describe polarization measurements and their correlations, it is sufficient to work in the two-dimensional plane transverse to the propagation direction. In this plane, a linear polarization state is specified by an **axis**, not a vector with orientation sign, and is therefore naturally represented by a unit direction defined modulo π. We represent such a polarization axis by a unit vector

$$\mathbf{a}(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \tag{3.4}$$

where α denotes the physical orientation angle of the polarizer. A rotation of the polarization axis by an angle α is then described by the standard two-dimensional rotation matrix

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \mathbf{a}(\alpha) = R(\alpha)\, \mathbf{a}(0). \tag{3.5}$$

Within this representation, the **probability amplitude** for a photon prepared with polarization axis $\mathbf{a}(\alpha)$ to pass a polarizer oriented along $\mathbf{a}(\beta)$ is given by the scalar overlap

$$\psi(\alpha, \beta) = \mathbf{a}(\alpha) \cdot \mathbf{a}(\beta) = \cos(\alpha - \beta). \tag{3.6}$$

The corresponding transmission probability is therefore

$$P(\alpha, \beta) = |\, \psi(\alpha, \beta) \,|^2 = \cos^2(\alpha - \beta), \tag{3.7}$$

which is precisely Malus' law.

3.4 Phase accumulation

The quantity $\psi(\alpha, \beta)$ carries not only magnitude but also **relative phase information**, encoded here geometrically through the relative rotation $\alpha - \beta$. While this phase plays no role in single-polarizer probabilities, it becomes essential when amplitudes associated with different independent measurement settings are combined prior to probability assignment, as in interference and entanglement scenarios.

Because polarization axes are defined modulo π, the squared amplitude of [3.3] can be written as

$$\cos^2(\alpha - \beta) = \frac{1}{2}[1 + \cos(2(\alpha - \beta))]. \qquad (3.8)$$

This **double-angle dependence** reflects the axial (rather than vectorial) nature of polarization and will be central in the derivation of bipartite correlation functions later.

Interpretational remark

In this formulation, phase is not introduced as an abstract complex scalar but as a **geometric consequence of relative rotations in the polarization plane**. The use of a squared amplitude reflects the bilinear character of measurement: a detection event depends on the compatibility between the photon's polarization axis and the analyzer orientation.

3.5 Relation to amplitude distributions

As in the Stern–Gerlach case, one may formally associate polarization preparation with a planar amplitude structure that includes both positive and negative contributions. However, for polarization filtering the projection formalism (3.2) – (3.8) is mathematically exact and operationally complete. The essential physics lies in the linear action of the analyzer on amplitudes and the subsequent formation of probabilities from squared overlaps.

3.6 Summary of polarization filtering

We have shown that:

1. Linear polarization analyzers act as geometric projectors for particle amplitudes.
2. Malus' law follows directly from the geometric overlap of polarization axes.
3. Phase tracking provides a compact and explicit description of polarization rotation.

4. The characteristic double-angle dependence arises from the amplitude-to-probability mapping and the axis nature of linear polarization.

Classical descriptions of polarization in terms of Stokes parameters [6] may be recovered as expectation values of the same underlying amplitude geometry.

In the next section photon polarization provides a direct two-dimensional realization of this phase-consistent amplitude framework, introducing the coherence condition.

4. Two-Photon Phase Coherence and Vector Amplitude Representation

In the preceding sections, spin and polarization filtering were treated as local operations acting on single-particle amplitudes. We now extend this framework to bipartite systems. The essential additional ingredient is that entangled pairs share a phase-coherent amplitude structure established at a common source.

A key point is that polarization amplitudes are not fully characterized by a single angular variable. Each amplitude corresponds to a vector in the transverse plane, and its phase information is encoded in its orientation. A complete description therefore requires tracking the amplitude as a geometric vector rather than reducing it to a scalar function of angle alone.

4.1 Shared amplitude structure at the source

Consider a source emitting two photons in opposite directions along the z-axis. The polarization of each photon lies in the transverse plane and is represented by a unit vector

$$\mathrm{a}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{4.1}$$

where θ is defined modulo π, reflecting the axial nature of polarization.

For an entangled pair, the preparation imposes a shared phase reference. Both photons are described by the same amplitude vector $\mathrm{a}(\theta)$, where θ is distributed uniformly over the interval $[0, \pi]$. This distribution reflects rotational invariance of the source.

4.2 Local analyzer projections

A linear polarizer oriented at angle $\alpha$is represented by the unit vector

$$\mathrm{e}(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}. \tag{4.2}$$

The transmission amplitude for a photon with internal amplitude $\mathrm{a}(\theta)$is given by the scalar projection

$$\psi_A(\alpha,\theta) = \mathrm{e}(\alpha)\cdot\mathrm{a}(\theta) = \cos(\alpha-\theta). \tag{4.3}$$

Similarly, for the second photon measured at angle β,

$$\psi_B(\beta,\theta) = \cos(\beta-\theta). \tag{4.4}$$

These are strictly local operations: each analyzer acts only on the amplitude vector arriving at its location.

4.3 Joint amplitude structure and correlation

Because the two photons share the same underlying phase parameter θ, the local responses at the analyzers are not independent but are evaluated on a common amplitude structure.

For linear polarization, the physically relevant observables are axial, meaning that directions differing by $\pi$are equivalent. This implies that the natural measurement responses are functions of the doubled angle. We therefore define the local response functions as

$$X(\alpha,\theta)=\cos\big(2(\alpha-\theta)\big), Y(\alpha,\theta)=\sin\big(2(\alpha-\theta)\big), \tag{4.5}$$

and similarly for the second analyzer at angle β,

$$X(\beta,\theta)=\cos\big(2(\beta-\theta)\big), Y(\beta,\theta)=\sin\big(2(\beta-\theta)\big). \tag{4.6}$$

These two components represent the complete phase-coherent amplitude structure of the polarization state in the transverse plane.

4.4 Correlation as phase-coherent bilinear pairing

The observable correlation is defined as the average of the bilinear pairing of these local responses over the shared phase distribution. Because the amplitude structure resides in a two-dimensional transverse plane, a complete geometric pairing between the two local responses requires the inner product of both phase quadratures of the amplitude vectors, encompassing both the aligned (X) and orthogonal (Y) quadratures. Both photons carry the same phase parameter θ, and the correlation function is

$$E(\alpha,\beta)=\frac{1}{\pi}\int_0^{\pi}[X(\alpha,\theta)\,X(\beta,\theta)+Y(\alpha,\theta)\,Y(\beta,\theta)]\,d\theta. \tag{4.7}$$

The prefactor $1/\pi$ ensures normalization over the interval $[0,\pi]$. Substituting the definitions (4.5)(4.6), the integrand becomes

$$\cos\big(2(\alpha-\theta)\big)\cos\big(2(\beta-\theta)\big)+\sin\big(2(\alpha-\theta)\big)\sin\big(2(\beta-\theta)\big). \tag{4.8}$$

Using the trigonometric identity

$$\cos\,u\cos\,v+\sin\,u\sin\,v=\cos\,(u-v),$$

with

$$u=2(\alpha-\theta), v=2(\beta-\theta),$$

the integrand simplifies to

$$\cos\big(2(\alpha-\beta)\big), \tag{4.9}$$

which is independent of θ.

4.5 Explicit evaluation

Because the integrand is constant with respect to θ, the integral evaluates immediately:

$$E(\alpha,\beta) = \frac{1}{\pi}\int_0^\pi \cos\big(2(\alpha-\beta)\big)\ d\theta = \cos\big(2(\alpha-\beta)\big). \tag{4.10}$$

This is precisely the standard quantum correlation function for polarization-entangled photon pairs.

Interpretation

This derivation shows that the correlation structure arises from:

1. Shared phase at preparation:
   The two photons carry a common phase parameter θ established at the source.
2. Local analyzer projections:
   Each measurement corresponds to a geometric projection acting only on the local amplitude.
3. Complete phase tracking:
   Both quadratures of the phase structure (cosine and sine components) are retained and combined before probability assignment.

The appearance of the double-angle dependence reflects the axial nature of polarization, while the absence of any residual dependence on θ shows that the correlation is determined entirely by the relative analyzer orientation.

Thus, the observed correlation emerges from phase-coherent amplitude structure carried from the source and processed locally at the analyzers, without requiring any dynamical interaction between spatially separated measurement events.

While this continuous geometric pairing captures the full kinematic structure of the correlation, the identical result is recovered through strict operational probability assignment across the four discrete measurement channels, as demonstrated in Appendix C.

5. Validation: Saturation of the Tsirelson Bound

In Section 4, we demonstrated that modeling bipartite measurements as geometric projections of phase-coherent amplitudes preserves amplitude information that naturally yields the exact correlation function eq.(4.10) $E(\alpha,\beta) = \cos\big(2(\alpha-\beta)\big)$ for polarization-entangled photons. Crucially, this result was obtained by carrying forward an enriched amplitude description without assuming dynamical influence between spatially separated measurement events; it emerges entirely locally from the retention of continuous geometric phase information prior to the formulation of classical probabilities.

5.1 Kinematic Implications of Geometric Projections

Because the amplitude-level formulation retains the full phase structure of the quantum state, the resulting correlation function is mathematically identical to the standard quantum prediction and therefore saturates the Tsirelson bound [3], reaching the theoretical maximum for quantum correlations ($S = 2\sqrt{2}$).

As a test of Bell's inequality [7], the Clauser-Horne-Shimony-Holt (CHSH) [8] measurement scenario serves as the benchmark for evaluating bipartite correlations. Having derived the foundational correlation function eq. (4.10) from continuous geometric overlaps, we have arrived at the exact mathematical starting point utilized in standard quantum mechanical proofs.

Because our geometric framework outputs a correlation function mathematically identical to the established quantum prediction, applying the CHSH summation for the standard sequence of optimized measurement angles is a straightforward replication of established quantum derivations. The explicit execution of the CHSH sum—along with a rigorous algebraic proof mapping this local phase geometry to the non-commuting components of the Pauli operator algebra is shown in Appendix D.

5.2 Enriched bipartite amplitude tracking versus probability-level factorization

Historically, the inability of classical local hidden-variable (LHV) models to surpass the Bell limit of $|S| \leq 2$ has been widely claimed as definitive proof of quantum non-locality. However, classical LHV models are constrained by a specific mathematical requirement: they strictly enforce probability-level factorization ($P_{AB} = P_A \times P_B$). This constraint acts as a kinematic coarse-graining that prematurely destroys the internal geometric phase structure of the individual particles before the correlation can be properly evaluated. It represents a method where system information is lost.

By contrast, when the measurement process is modeled using the full continuous geometry of the particle—combining phase-resolved amplitudes bilinearly ***prior*** to macroscopic probability assignment—the $2\sqrt{2}$ expectation emerges natively. Because this framework reproduces the Tsirelson bound through local projection operations on a shared preparation geometry, it provides a geometrically transparent representation of the standard quantum result. We conclude that this supports the adequacy of our model for enriched amplitude analysis and the way we track bipartite amplitude propagation.

Within the standard quantum formalism, the departure from the classical Bell bound originates from the noncommuting structure of rotated observables. In the present formulation, this structure appears explicitly as phase coherence at the amplitude level.

Our formulation also differs slightly from the traditional Einstein–Podolsky–Rosen perspective [9] by treating phase coherence as a physically retained structure rather than a deficiency of the quantum description. We enrich the quantum description with a higher degree of accuracy than binary spin provides. We do not disagree with Bell's theorem per se, we only demonstrate that it does not restrict all types of information carried forward from locally prepared entangled photons.

6. Interpretational Remarks

This section summarizes what has been established and clarifies what is—and is not—being claimed.

6.1 What the analysis does

The paper constructs a phase-tracked description of preparation, rotation, filtering, and correlation for (i) spin-½ particles in Stern–Gerlach experiments and (ii) photon polarization in linear analyzers. The essential steps are:

1. **Post-selection induces structured directional amplitude densities.**
   In the Stern–Gerlach case, post-selection of a channel is represented by each particle being given a directional amplitude within a directional amplitude density on S^2 of the form (2.1). This object is normalized but not pointwise nonnegative everywhere, and it is not interpreted as a classical probability density over hidden directions. Its negative regions are essential for reproducing the $\cos^2(\alpha/2)$ rotation law from hemisphere integration.

2. **Filtering is amplitude projection on analyzers.**
   For polarization, the transmission probability is obtained from the squared overlap of transverse polarization axes, yielding Malus' law (3.3). The double-angle dependence arises from the amplitude-to-probability mapping and the axial nature of linear polarization.

3. **Entanglement is treated as a relational phase constraint brought forward.**
   Two-photon entanglement is represented by phase coherence between joint amplitudes established at the source and preserved under free propagation. Separated analyzers interact independently with each photon's amplitude. The correlation emerges because joint probabilities depend on complex interference terms that survive when relative phase is tracked consistently and the probability is calculated prior to measurement at the analyzer.

4. **CHSH violations are traced to noncommutativity (phase structure).**
   In the Pauli-algebra operator analysis in Appendix D, the CHSH departure from the Bell bound is controlled by the commutators $[A, A']$ and $[B, B']$, which are noncommuting objects encoding local phase structure. The Tsirelson bound follows as a norm bound, without invoking any superluminal dynamics.

6.2 What "locality" means here

The term "locality" is used in the operational, relativistic sense:

- The outcome at Alice's station is solely produced by interactions within Alice's apparatus region.
- The outcome at Bob's station is solely produced by interactions within Bob's apparatus region.
- No signal, force, or causal influence is required to propagate between stations at measurement time.

The analysis is therefore local at the level of **physical interactions and dynamical influence**. Correlations are attributed to common preparation and phase-coherent amplitude combination locally at the origin of an entangled pair without introducing any dynamical influence between spatially separated measurement events.

6.3 What is not assumed.

The analysis does not assume that all potential measurement outcomes can be represented simultaneously as functions of a single classical random variable equipped with a fixed, nonnegative probability distribution $\rho(\lambda)$ from which correlations are obtained by probability-level averaging.

Instead, outcomes are computed from a shared, phase-resolved preparation structure using amplitude-level relations. The framework retains the defining quantum feature that **interference occurs prior to probability assignment**. Noncommuting phase components contribute before scalar extraction.

This difference is already visible in the Stern–Gerlach treatment: reproducing the correct rotation law required a signed density (2.1). Such objects do not fall within the classical probability measures typically assumed in Bell-type derivations.

6.4 Realism and what is meant by "amplitude structure"

The framework is compatible with a modest form of local realism:

- preparation devices imprint a structured amplitude relation on outgoing particles,
- this structure maintains its phase-dependency as it propagates,
- separated measurement devices perform local projections that yield discrete outcomes,
- observed statistics arise from repeated trials with stable preparation and stable local response.

However, the analysis does not rely on “local realism” in the strong sense of assigning definite outcomes to counterfactual measurements. Instead, it requires only that relative phase, established at preparation, be treated as a physically relevant relational attribute for computing correlations. All outcomes remain locally generated, while phase coherence is retained at the level necessary to reproduce the observed quantum correlations.

6.5 Interpretational Remark

The present work does not modify the formalism of quantum mechanics, which already incorporates phase information through complex amplitudes or, equivalently, geometric rotation formalism. Instead, it provides a geometrically explicit representation of how phase coherence enters Bell-type correlation calculations within the standard formalism. When phase is treated as an intrinsic component of joint amplitudes rather than as an auxiliary or unphysical quantity, the quantum correlation structure is recovered without abandoning locality at the level of physical interactions. From this perspective, entanglement reflects a relational phase constraint encoded in the joint initial state, not a dynamical influence propagating instantly between spatially separated systems. The results demonstrate that the experimentally observed Bell/CHSH correlations can be fully recovered through phase-coherent amplitude algebra, without requiring superluminal influences of any kind.

7. Conclusion

This work does not extend quantum mechanics but reformulates its amplitude structure in a geometrically explicit way using signed directional amplitude densities. Within this representation, the emergence of Tsirelson saturation appears directly from phase-coherent local projections.

Spin and polarization correlations are re-examined from the perspective of phase-tracked amplitude structure. Rather than introducing hidden variables or modifying quantum dynamics, we focus on how physical preparation devices—such as Stern–Gerlach analyzers and polarization filters—induce structured directional amplitude distributions $\rho(\theta)$ that retain phase coherence under rotation. We identify such distribution (2.1) and normalize it. Identifying the amplitude properties of particles and their distribution ρ is the decisive first step that enables us to track the common phase of entangled particles.

For photon polarization we derived Malus’ law directly from geometric projection in the transverse plane and expressed polarization rotations with planar rotation operators, clarifying the origin of the double-angle dependence.

Extending the framework to entangled photon pairs, we treated entanglement as a relational joint phase, θ, constraint on amplitudes prepared at the source and preserved under free propagation. Separated analyzers act independently on each photon's amplitude; probability outcome arise at the individual analyzer based on amplitude interference prior to probability assignment locally, while applying the expected structured post-analyzer distribution.

In the CHSH setting, we showed that the Tsirelson bound follows from the noncommuting phase components of geometrically rotated observables, making explicit that violation of the Bell bound is controlled by local phase structure rather than by any dynamical influence between spatially separated measurement events.

Bell's theorem remains a valid constraint on probability-only factorizations. The present work instead identifies a phase-coherent directional amplitude mechanism whose correlations lie outside probability-only hidden-variable factorization. This provides a consistent interpretation of quantum correlations in which entanglement encodes relational phase structure rather than instantaneous action at a distance.

The non-commuting components of local observables encode phase information associated with rotations which we can represent explicitly. The phase-resolved, directional amplitude description provides a natural framework for understanding filtering, sequential measurements, polarization phenomena, and interference within a unified local setting. This does not change anything in quantum theory; its contribution comes with offering an alternative organizational perspective on the amplitude structure already present in quantum mechanics.

Appendix A. Symmetry, Uniqueness, and the Stern-Gerlach Hemisphere Integral

This appendix establishes the mathematical uniqueness of the directional amplitude distribution associated with post-selection, and demonstrates its explicit evaluation, yielding the explicit mathematical reasoning behind the Stern–Gerlach transition probability used in Section 2. We closely follow standard treatments of spin rotation and state preparation found in the early spin literature and in modern pedagogical analyses of quantum mechanics, particularly those of Pauli [10] and Feynman [2]. The derivation shows that no alternative angular dependence is compatible with the imposed symmetry and operational constraints.

A.1 Axisymmetric Amplitude Distributions and Uniqueness

Consider a Stern-Gerlach apparatus oriented along a reference axis z. An initially isotropic beam passes through the device, and we post-select the "spin-up" output channel. We represent the prepared ensemble by a real, normalized directional amplitude density $\rho(\hat{s})$ on the unit sphere S^2.

Because the preparation singles out the axis z but is otherwise invariant under rotations about that axis, the density may depend only on the polar angle θ between the amplitude direction and z, or equivalently on $\mu = \cos\theta$. Any square-integrable axisymmetric function on the sphere admits a Legendre expansion:

$$\rho(\mu) = \sum_{l=0}^{\infty} a_l P_l(\mu). \tag{A.1}$$

A second analyzer oriented at an angle β transmits components based on integration over a hemisphere (derived explicitly in A.3). Because hemisphere integration is a rotation-covariant linear functional, it cannot mix distinct irreducible representations of the rotation group. Each Legendre component $P_l(\mu)$ in ρ contributes independently to the angular dependence of the transition probability $P(\beta)$.

Empirically, the standard transition probability for sequential spin-½ measurements is $P(\beta) = \cos^2\left(\frac{\beta}{2}\right) = (1 + cos\beta)/2$. This angular dependence contains only a constant term ($l = 0$) and a term proportional to $cos\beta = P_1(cos\beta)$.

This strictly implies that the prepared density $\rho(\mu)$ cannot contain any Legendre components beyond $l = 1$. If any coefficient a_l with $l \geq 2$ were nonzero, the corresponding higher-order harmonic would necessarily appear in $P(\beta)$, contradicting the observed form for arbitrary analyzer orientations. Thus, the most general admissible density is restricted to $\rho(\mu) = a_0 + a_1\mu$.

Normalization over the sphere fixes the constant term to $a_0 = 1/(4\pi)$. The remaining coefficient is fixed by the requirement that a second analyzer aligned with the preparation axis transmits with unit probability $P(0) = 1$, which yields $a_1 = 1/(2\pi)$.

The unique post-selection density is therefore forced to be:

$$\rho(\theta) = \frac{1}{4\pi} + \frac{1}{2\pi}\cos\theta. \tag{A.2}$$

This density is normalized over the full sphere, but it is not everywhere positive. As emphasized in the main text, $\rho(\theta)$ is not a classical probability distribution. It is a phase-weighted directional amplitude structure, and negative values encode destructive interference prior to probability assignment.

A.2 Second analyzer and hemisphere geometry

Now introduce a second Stern–Gerlach analyzer with axis $\hat{n}$, tilted by an angle α relative to $\hat{z}$.

The second analyzer transmits those components whose projection along $\hat{n}$ is positive. Geometrically, this corresponds to a hemisphere of directions: all unit vectors whose dot product with $\hat{n}$ is non-negative. A single spin-½ fermion will be assigned with spin up if its individual directional amplitude points less than 90 degrees off the $\hat{z}$-up channel of the Stern-Gerlach analyzer. More than 90 degrees off means spin down.

The transition probability for an ensemble of particles prepared in the $\hat{z}$-up channel to be transmitted by the $\hat{n}$-oriented analyzer is obtained by integrating the amplitude density over this hemisphere.

A.3 Evaluating the integral

The integral splits naturally into two parts.

1. Constant term

The integral of the constant part $1/(4\pi)$over a hemisphere is simply one half. This reflects the fact that a hemisphere covers half of the unit sphere.

2. Cosine term

The second term involves integrating $\cos\theta$ over the hemisphere defined by $\hat{n}$.

Geometrically, $\cos\theta$ is the dot product between the direction vector and $\hat{z}$. When integrated over the hemisphere defined by $\hat{n}$, symmetry arguments show that the result must be proportional to $\cos\alpha$, the projection of $\hat{n}$ onto $\hat{z}$.

A direct calculation yields:

$$\int_{\text{hemisphere}} \cos\theta \, d\Omega = \pi \cos\alpha \,. \qquad (\text{A}.3)$$

Including the prefactor $1/(2\pi)$, this contribution becomes $\frac{1}{2}\cos\ \alpha$.

A.4 Final result

Adding both contributions, the total transition probability is

$$P(\hat{z}\uparrow\rightarrow\hat{n}\uparrow) = \frac{1}{2} + \frac{1}{2}\cos\alpha = cos^2\left(\frac{\alpha}{2}\right). \qquad (A.4)$$

This is exactly the standard quantum-mechanical prediction for sequential Stern–Gerlach measurements on a spin-½ particle. A key point is that this result follows from a **local geometric integration of a directional amplitude structure for each particle**, without introducing any dynamical influence between spatially separated measurement events. The use of a signed density is essential for the correct outcome.

Appendix A.5 — Spin-½ Rotation Amplitudes (Feynman Convention)

Following the Euler-angle convention used by Feynman [2], the rotation from an initial spin axis S to a final axis T is described by angles (α, β, γ), corresponding to rotations α: z to z', then β: x to x_1, then γ: x_1 to x', where x_1 is an intermediate position

The transition amplitudes are

$$\langle +T \mid +S \rangle = \cos\frac{\alpha}{2}\ e^{\frac{i}{2}(\beta+\gamma)}$$

$$\langle +T \mid -S \rangle = i\sin\frac{\alpha}{2}\ e^{-\frac{i}{2}(\beta-\gamma)}$$

$$\langle -T \mid +S \rangle = i\ \sin\frac{\alpha}{2}\ e^{\frac{i}{2}(\beta-\gamma)}$$

$$\langle -T \mid -S \rangle = \cos\frac{\alpha}{2}\ e^{-\frac{i}{2}(\beta+\gamma)}$$

The present form follows Feynman's explicit Pauli-matrix convention, in which the imaginary unit appears in the off-diagonal elements. Different but equivalent representations are commonly used in the literature.

The corresponding transition probabilities are obtained by squaring the moduli of these amplitudes, yielding the familiar $\cos^2(\alpha/2)$ and $\sin^2(\alpha/2)$ laws.

These expressions show explicitly that spin transitions depend on rotation geometry and phase, not merely on relative orientation angles. The phase factors become essential when amplitudes are combined prior to probability assignment, as in interference and entanglement scenarios discussed in the main text.

Appendix B. Justification of the double-angle distribution

The choice

$$\rho(\theta) = \frac{1}{\pi}\cos(2\theta) \qquad (B.1)$$

is not arbitrary. Linear polarization is an **axial** degree of freedom: the physical state is invariant under $\theta \mapsto \theta + \pi$. Any amplitude distribution with this symmetry admits a Fourier expansion

$$\rho(\theta) = \sum_{n=0}^{\infty} c_n \cos(2n\theta). \qquad (\mathrm{B}.2)$$

The constant term c_0 contributes no correlations. To reproduce a pure $\cos\ 2(\alpha - \beta)$ dependence, the shared amplitude kernel must contain only the $n = 1$ axial mode; additional modes would introduce higher-frequency dependence in $\alpha - \beta$. Thus, the lowest nontrivial axial mode $n = 1$ is uniquely selected by symmetry.

For a normalized distribution one would require $\int \rho(\theta)d\theta = 1$; however, in the present amplitude-kernel formulation the $n = 1$ mode has zero mean over [0,π], and the prefactor 1/π is fixed by orthogonality scaling.

$$\int_0^{\pi} \rho(\theta)\, d\theta = 0, \qquad (B.3)$$

and since $\int_0^{\pi} \cos(2\theta)\, d\theta = 0$, the orthogonality scaling factor must be $1/\pi$.

Symmetry remark

The double-angle structure reflects the fact that linear polarization is an axial quantity: θ and $\theta + \pi$ represent the same physical direction. The distribution $\rho(\theta)$ is therefore the lowest-order axial Fourier mode consistent with rotational symmetry and the observed correlation function.

Appendix C. Operational Recovery of the Correlation Function

This appendix provides an operational derivation of the bipartite correlation function. While Section 4 establishes that the full correlation natively emerges from the continuous geometric pairing of complete amplitude vectors in the transverse plane, we demonstrate here that the exact same result is recovered through strict discrete probability assignment across the four measurement channels. This confirms that the continuous geometric framework is fully consistent with standard operational measurement constraints.

C.1 The Four Discrete Measurement Channels

In a standard bipartite Bell-type experiment, two spatially separated observers perform local polarization measurements at angles α and β. Each measurement yields a binary outcome (+1 for transmission, –1 for absorption/orthogonal transmission). There are four possible joint outcomes: (+ +), (– –), (+ –), and (– +).

For an entangled pair sharing a common geometric orientation (a singlet-type correlation in the transverse plane), the source is unpolarized, meaning the initial orientation is uniformly distributed. The probability of both photons being transmitted (+ +) is determined by the unpolarized source probability (1/2) multiplied by the geometric transmission probability dictated by Malus' law for the relative angle:

$$P(++) = \left(\frac{1}{2}\right) cos^2(\alpha - \beta) \qquad (C.1)$$

By symmetry, the probability that both photons are absorbed (or transmitted into the orthogonal channels) is identical:

$$P(--) = \left(\frac{1}{2}\right) cos^2(\alpha - \beta) \qquad (C.2)$$

For the orthogonal cross-channels (+ –) and (– +), the effective measurement axis for the second photon is rotated by $\pi/2$ relative to the first (i.e., the relative angle is α – (β + π/2)). The geometric overlap becomes $cos(\alpha - \beta - \pi/2) = sin(\alpha - \beta)$. Squaring this amplitude yields the cross-channel probabilities:

$$P(\pm) = \left(\frac{1}{2}\right) sin^2(\alpha - \beta) \quad P(\mp) = \left(\frac{1}{2}\right) sin^2(\alpha - \beta) \tag{C.3}$$

C.2 Expectation Value and the Double Angle The observable correlation function, $E(\alpha, \beta)$, is defined operationally as the expectation value of the joint outcomes. It is computed by subtracting the probabilities of the anti-correlated outcomes from the correlated outcomes:

$$E(\alpha, \beta) = [P(+\,+) + P(-\,-)] - [P(\pm) + P(\mp)] \tag{C.4}$$

Substituting the geometric probabilities derived above:

$$E(\alpha, \beta) = \left[\left(\frac{1}{2}\right) cos^2(\alpha - \beta) + \left(\frac{1}{2}\right) cos^2(\alpha - \beta)\right] - \left[\left(\frac{1}{2}\right) sin^2(\alpha - \beta) + \left(\frac{1}{2}\right) sin^2(\alpha - \beta)\right] \tag{C.5}$$

This simplifies algebraically to:

$$E(\alpha, \beta) = cos^2(\alpha - \beta) - sin^2(\alpha - \beta) \tag{C.6}$$

Using the standard trigonometric double-angle identity ($cos^2 x - sin^2 x = cos\ 2x$), this expression resolves perfectly to:

$$E(\alpha, \beta) = cos\big(2(\alpha - \beta)\big) \tag{C.7}$$

3 Conclusion The discrete operational calculation yields the exact same double-angle cosine correlation derived via the continuous vector pairing in Section 4. This demonstrates that the geometric amplitude framework does not rely on measuring mutually exclusive quadratures simultaneously. Instead, applying standard probability assignment to the local geometric overlaps accurately reproduces the full quantum correlation, confirming the operational validity of the approach.

Appendix D. CHSH Correlations and Phase-Consistent Locality

This section connects the phase-coherent amplitude framework to the CHSH scenario. We (i) define the CHSH correlation function for binary outcomes, (ii) derive the quantum / Tsirelson bound using a Pauli-algebra [10]) operator identity that makes the role of phase explicit per entangled pair of photons, and (iii) state precisely how Bell-type probabilistic assumptions differ from phase-coherent amplitude constructions.

Bell's theorem [7] establishes bounds on correlations constructed from probability-level factorizations of local outcomes, under assumptions that exclude noncommuting phase structure.

Throughout, "locality" is taken in the operational sense that each measurement outcome is produced by a local interaction between the photon (the particle) and the analyzer, with no superluminal dynamical influence involved between entangled pairs. A **common initial phase reference** established at emission is required and a common preparation geometry (opposite directions, same phase, etc.). By assigning a specific amplitude distribution per particle we are able to follow an entangled pair of photons with their correlated phases still intact at the polarizers. And we know their post-analyzer statistical distribution.

D.1 CHSH setting and correlators

The CHSH inequality was introduced in its experimentally testable form by J. Clauser et al [8], and subsequently tested in landmark experiments by Alain Aspect [11].

Let Alice choose between two analyzer settings $a, a'$and Bob between b, b'. Each run yields binary outcomes

$$A(a), A(a') \in \{+1, -1\}, \qquad B(b), B(b') \in \{+1, -1\}. \tag{D.1}$$

Define the correlation function

$$E(a, b) = \langle A(a)\, B(b) \rangle, \tag{D.2}$$

where $\langle \cdot \rangle$denotes the ensemble mean over many trials. The CHSH combination is

$$S \;=\; E(a, b) + E(a, b') + E(a', b) - E(a', b'). \tag{D.3}$$

In classical local-hidden-variable (LHV) models one typically assumes the existence of a probability space (λ, ρ) and deterministic (or stochastic) response functions $A(a, \lambda)$, $B(b, \lambda)$ such that

$$E_{\text{LHV}}(a, b) = \int_{\lambda} \rho(\lambda)\, A(a, \lambda)\, B(b, \lambda)\, d\lambda. \tag{D.4}$$

Under these assumptions, $| S | \leq 2$follows. See section C.6-C.7.

In the present work, by contrast, correlations are generated from geometric **phase-coherent amplitude algebra** (Section 4), in which **interference occurs prior to probability assignment**. As a result, the probabilistic form (D.4) does **not** represent the most general description consistent with phase tracking. We will make this distinction explicit after establishing the operator bound.

D.2 Operator formulation in Pauli formalism

To express the role of noncommuting phase structure cleanly, we use the standard Pauli matrix algebra [12] for dichotomic spin observables. For a unit vector **a**, define the corresponding observable

$$A \equiv \sigma_{\mathbf{a}}\,, \;\; A' \equiv \sigma_{\mathbf{a}'}\,, \;\; B \equiv \sigma_{\mathbf{b}}\,, \;\; B' \equiv \sigma_{\mathbf{b}'}. \tag{D.5}$$

These satisfy

$$A^2 = (A')^2 = B^2 = (B')^2 = 1. \tag{D.6}$$

The geometric product implies the Pauli-matrix identity

$$\sigma_{\mathbf{a}}\sigma_{\mathbf{a}'} = \mathbf{a}\cdot\mathbf{a}' + i(\mathbf{a}\times\mathbf{a}'), \tag{D.7}$$

where i is the imaginary unit ($i^2 = -1$). Consequently,

$$\{\sigma_{\mathbf{a}}, \sigma_{\mathbf{a}'}\} = 2\,\mathbf{a}\cdot\mathbf{a}', \qquad [\sigma_{\mathbf{a}}, \sigma_{\mathbf{a}'}] = 2\,i(\mathbf{a}\times\mathbf{a}'). \tag{D.8}$$

The commutator term constitutes the explicit "phase-like" noncommuting component.

For bipartite measurements, we assume the usual locality of operator algebras: Alice's and Bob's observables act on different subsystems and therefore commute:

$$[A, B] = [A, B'] = [A', B] = [A', B'] = 0. \tag{D.9}$$

Equation (D.9) expresses locality at the level of operator algebras: observables associated with distinct subsystems commute. This does not imply commutativity among different measurement choices at the same site, which is generally violated and plays a central role in the failure of Bell-type inequalities.

D.3 The CHSH operator and its square

Using the Pauli algebra identity derived above eq (D.7), the square of the CHSH operator reduces to

$$\boxed{\mathcal{S}^2 = 4 - [A, A']\,[B, B']}. \tag{D.10}$$

This is a central algebraic step: any departure from the classical Bell bound is controlled by the product of commutators — i.e. by the noncommuting content of rotated observables.

D.4 Bounding the norm: Tsirelson bound

From (D.10),

$$\| \mathcal{S}^2 \| \le 4 + \| [A, A'] \| \ \ \| [B, B'] \|. \tag{D.11}$$

Using (D.8),

$$[A, A'] = 2\,i(\mathbf{a}\times\mathbf{a}'), \ \ [B, B'] = 2\,i(\mathbf{b}\times\mathbf{b}'). \tag{D.12}$$

Since $|\ \mathbf{a}\times\mathbf{a}'\ | = \sin\ \theta_A$ where θ_A is the angle between $\mathbf{a}$ and $\mathbf{a}'$,

$$\| [A, A'] \| = 2\ |\ \sin\ \theta_A\ | \le 2, \| [B, B'] \| = 2\ |\ \sin\ \theta_B\ | \le 2. \tag{D.13}$$

Therefore

$$\| \mathcal{S}^2 \| \leq 4 + (2)(2) = 8, \Rightarrow \boxed{\| \mathcal{S} \| \leq 2\sqrt{2}}. \tag{D.14}$$

Finally, by (D.11),

$$\boxed{| S | \leq 2\sqrt{2}}. \tag{D.15}$$

This is the Tsirelson bound.

Equality is achieved when the local measurement settings are mutually noncommuting, i.e. when the commutators in (D.10) attain their maximal magnitude. For photon polarization, this corresponds to a 45° separation between the physical analyzer orientations at each station, reflecting the double-angle structure of linear polarization. In the polarization case, a 45° separation of analyzer axes corresponds to orthogonal observables in the underlying SU(2) operator algebra, which is why the same Tsirelson bound is obtained.

D.5 Explicit CHSH Evaluation

Having established the phase-coherent correlation function

$$E(\alpha, \beta) = \cos\ 2(\alpha - \beta) \tag{D.16}$$

strictly from local amplitude overlaps in Section 4, we evaluate it using the standard sequence of measurement angles known to maximize quantum correlations: $a = 0°$, $a' = 45°$, $b = 22.5°$, $b' = -22.5°$.

Substituting these angles into the locally derived correlation function yields:

$$E(a, b) = cos\ 2(0^o - 22,5^o) = \sqrt{2}/2$$

$$E(a, b') = cos\ 2(0^o - (-22,5^o)) = \sqrt{2}/2$$

$$E(a', b) = cos\ 2(45^o - 22,5^o) = \sqrt{2}/2$$

$$E(a', b') = cos\ 2(45^o - (-22,5^o)) = -\sqrt{2}/2$$

Summing these expectations into the CHSH parameter S:

$$S = E(a, b) + E(a, b') + E(a', b) - E(a', b') = 2\sqrt{2} \qquad (D.17)$$

This explicit evaluation confirms that the phase-coherent amplitude framework, despite relying entirely on local geometry and local projections, mathematically reproduces the maximum violation of the Bell inequality and exactly saturates the Tsirelson bound.

D.6 Phase-consistent locality and the scope of Bell-type inequalities

Bell's inequality derivations constrain models of the form (D.4), where:

1. The correlation is built from products of locally assigned outcomes $A(a, \lambda)$, $B(b, \lambda)$.
2. Averaging is performed over a **single** nonnegative probability measure $\rho(\lambda)$.
3. Any "hidden description" λ functions as a classical random variable, i.e. it carries no phase structure.

By construction, the phase-coherent amplitude framework used here is not a model of type (D.4). The essential difference is the placement of phase and interference:

- In (D.4), averaging is performed on **outcome products** (probabilities-first).
- In the phase-coherent framework, amplitudes carry phase structure and are combined **prior** to forming probabilities (amplitudes-first).

This distinction is already visible at the single-particle level in Section 2: reproducing Stern-Gerlach transition probabilities required a signed (quasi-)density, not a nonnegative classical probability density. The same conceptual point is reflected in the CHSH operator identity (D.10): the contribution beyond the classical bound is controlled by commutators, i.e. local noncommutativity (phase structure), rather than by probability-level averaging.

Accordingly, the present analysis does not dispute Bell's theorem within its domain of assumptions; rather, it exhibits a phase-consistent amplitude mechanism in which correlations are locally generated while lying outside probability-only hidden-variable factorization.

D.7 Commuting and Non-Commuting Observables

Bell-type inequalities are derived within a probabilistic framework in which measurement outcomes are represented by real-valued random variables. In such models, the outcome functions $A(a, \lambda)$ and $B(b, \lambda)$ associated with different measurement settings are assumed to take numerical values (typically ± 1) and to combine multiplicatively. As ordinary numbers, these quantities commute:

$$A(a, \lambda)B(b, \lambda) = B(b, \lambda)A(a, \lambda). \qquad (\mathrm{D.18})$$

This commutativity is not an additional physical assumption, but a structural property of classical probability theory, which averages products of commuting random variables. Consequently, **Bell-type derivations necessarily exclude any noncommuting algebraic structure at the level of amplitude phase correlations**.

By contrast, quantum observables associated with different measurement directions are represented by operators whose products generally contain noncommuting components. These components encode phase information associated with rotations

and are responsible for interference effects that **occur prior to probability assignment**. When such phase structure is retained, correlation functions are formed at the amplitude level, and only subsequently converted into probabilities.
The distinction is therefore not between “local” and “non-local” variables, but between **commuting outcome-based models** and **phase-coherent amplitude-based models**. Bell inequalities constrain the former class, while the latter lies outside their scope.

D.8 Summary

We have shown that:

1. The CHSH value is bounded by $2\sqrt{2}$ in our framework.
2. The operator identity $\mathcal{S}^2 = 4 - [A, A'] \, [B, B']$ makes explicit that the departure from the classical bound of 2 originates from local noncommutativity (phase structure), not from a dynamical influence between spatially separated measurement events.
3. For polarization entanglement, the experimentally observed $\cos\, 2(\alpha - \beta)$ correlation is recovered by local projection acting on phase-coherent joint amplitudes.
4. Bell inequalities constrain probability-only local models based on classical probability factorization methods; The presented phase-coherent amplitude models are not of that type; rather, this enriched amplitude framework provides a more detailed kinematic description than standard discrete spin labels, thereby illuminating the local physical processes that take place during entanglement.

We have demonstrated that an individual particle undergoing a binary measurement (such as in a Stern-Gerlach apparatus or a linear polarizer) can be kinematically described by a continuous directional amplitude, parameterized by its geometric deviation from the analyzer axis. A defining feature of this framework is that following a measurement, the sub-ensemble within a given transmission channel acquires a fixed, characteristic amplitude distribution about the new axis, entirely independent of its pre-measurement distribution. The act of measurement, therefore, acts as a geometric reset filter: selection into a specific channel rigorously prepares a known, phase-coherent post-analyzer directional amplitude distribution $\rho(\theta)$. Exploring the broader theoretical consequences of this amplitude distribution is left for future work.

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