Geometric Marginalization and the Uniqueness of Phase-Coherent Amplitude Distributions in Quantum Measurement

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Abstract

In a recent work, we established a phase-coherent directional amplitude framework that natively reproduces the full spectrum of bipartite quantum correlations, exactly saturating the Tsirelson bound $2\sqrt{2}$. While that work demonstrated the predictive power of retaining continuous geometric phase information, the structural origin and uniqueness of the underlying signed amplitude distributions remained implicit.

In the present paper we show that these distributions are uniquely determined by symmetry and measurement geometry. We propose a geometric interpretation of quantum measurement in which state reduction arises from marginalization over operationally inaccessible degrees of freedom. An arriving particle and a measurement apparatus uniquely determine a plane of relevance, while orthogonal degrees of freedom are physically filtered and marginalized from the effective description.

Using harmonic analysis on the sphere, we prove that this geometric marginalization—together with rotational covariance and normalization—uniquely fixes the post-selection amplitude distributions. For spin-½ systems, rotational symmetry restricts the Legendre expansion to its lowest harmonics, forcing the exact signed distribution previously employed. Extended to bipartite product spaces $S^2 \times S^2$, global rotational invariance uniquely determines the joint amplitude structure that reproduces the standard singlet correlation law.

These results provide a kinematic explanation of the amplitude distributions underlying quantum spin correlations and offer a geometric account of quantum state reduction as marginalization over degrees of freedom that are operationally inaccessible to the measurement interaction.

1. Introduction

Quantum mechanics provides an extraordinarily successful formalism for predicting measurement outcomes [1]. However, the physical interpretation of the measurement process itself—specifically the notion of state reduction or "collapse"—is often introduced as an operational postulate distinct from unitary dynamics. This raises persistent conceptual questions regarding what exactly is reduced during measurement, and what kinematic mechanisms drive this reduction.

In a recent paper [2], we addressed the kinematic foundations of quantum states by introducing a phase-resolved, directional amplitude description for individual spin and polarization degrees of freedom. By treating measurement as a geometric projection acting on these continuous amplitudes, it was shown that the retention of phase coherence prior to probability assignment natively reproduces the full quantum correlation statistics. When applied to entangled photon pairs, this phase-coherent framework naturally recovers the standard cosine correlation form and exactly saturates the Tsirelson bound ($2\sqrt{2}$) [3], demonstrating that the full phenomenology of bipartite correlations can be accounted for by the richer algebraic structure of continuous amplitudes.

While that work established the empirical adequacy of the framework, it left open a fundamental structural question: where do the specific directional amplitude distributions used to model these states come from, and why must they take the precise mathematical form they do?

The present paper addresses this gap. We propose that much of the ambiguity surrounding quantum measurement arises from a failure to distinguish between kinematic description and operational accessibility. We show that quantum measurement can be understood as a process of **geometric marginalization**. By establishing orientation as the primitive kinematic variable, we demonstrate that state reduction is not a dynamical process acting on reality, but a geometric filtering: it is the marginalization of degrees of freedom that are orthogonal to the plane uniquely defined by the particle and the measurement apparatus.

In this picture, the measurement update rule usually referred to as "collapse" is not an additional dynamical postulate. It arises naturally as geometric marginalization: once the interaction between particle and apparatus defines a plane of relevance, degrees of freedom orthogonal to that plane become operationally inaccessible and are integrated out of the effective description.

Furthermore, we provide the rigorous mathematical proofs underpinning this mechanism. We show that the post-selection distributions induced by this marginalization are uniquely fixed by rotational symmetry, normalization, and operational transition rules. These symmetry constraints restrict the admissible distributions to the minimal harmonic structure compatible with the observed transition probabilities.

2. Geometric Marginalization and State Reduction

For systems such as spin-½ particles [4] and polarized photons, the experimentally relevant degrees of freedom are inherently geometric. Measurement outcomes depend on angular relations between preparation and measurement settings [5]. We therefore represent the state of an individual particle, prior to measurement, by a unit vector in an abstract orientation space. For spin systems, this corresponds to a direction on the Bloch sphere; for polarization, to an axis in a transverse plane. This orientation specifies the directional structure of the particle's amplitude.

A measurement device introduces local structure by defining a preferred axis, $\hat{a}$. Given a particle with an incoming orientation $\hat{s}$ and a measurement axis $\hat{a}$, there exists a unique plane, $\boldsymbol{\Pi}(\hat{s}, \hat{a})$, spanned by these two vectors. This plane is fully determined by the geometry of the interaction.

Crucially, all scalar quantities relevant to the measurement—such as the projection $\hat{s} \cdot \hat{a}$—lie within this plane. Any components of a more detailed description that would distinguish directions orthogonal to $\boldsymbol{\Pi}(\hat{s}, \hat{a})$ cannot influence the measurement outcome, because the apparatus has no means of coupling to them.

At the analyzer, the interaction physically removes incompatible components of the incoming state. The transmitted particle emerges in a constrained but definite state consistent with the geometry of the filter. Once the relevant plane $\boldsymbol{\Pi}(\hat{s}, \hat{a})$ is identified, all degrees of freedom orthogonal to this plane are operationally inaccessible and must be excluded from the effective description of the post-measurement state.

This state reduction is the geometric marginalization of these now-inaccessible degrees of freedom. In probabilistic language, it corresponds to integrating out inaccessible variables; in amplitude-based language, it corresponds to discarding interference structures associated with those degrees of freedom.

We can physically interpret this as **phase-conditioned directional coarse-graining**. At the moment of interaction, the internal phase defines a transverse plane, one transverse degree of freedom is retained conditionally, and the orthogonal component is effectively erased through averaging. Collapse is therefore not a special dynamical event, but the transition from a full kinematic description to a reduced description conditioned entirely on the geometry of the measurement interaction.

3. Symmetry and Uniqueness of the Single-Particle Distribution

We now establish the mathematical uniqueness of the directional amplitude distribution associated with post-selection in a Stern-Gerlach [6] experiment, proving that geometric marginalization uniquely forces the distribution utilized in our previous bipartite analysis.

Let $\hat{s}$ denote a unit vector on the sphere S^2, representing the directional spin amplitude of an individual particle. After post-selection by an analyzer oriented along a fixed axis $\hat{z}$, the prepared ensemble is characterized by a real, normalized directional amplitude density $\rho(\hat{s})$ on S^2.

Because the preparation singles out the axis $\hat{z}$ but is otherwise invariant under rotations about that axis (as a result of marginalizing orthogonal degrees of freedom), ρ may depend only on the polar angle θ between $\hat{s}$ and $\hat{z}$, or equivalently on $\mu = cos\ \theta$. No dependence on the azimuthal angle is permitted.

Any square-integrable axisymmetric function on the sphere admits a Legendre expansion,

$$\rho(\mu) = \sum_{\ell=0}^{\infty} a_\ell \, P_\ell(\mu). \tag{3.1}$$

Because hemisphere integration is a rotation-covariant linear functional, it cannot mix distinct irreducible representations of the rotation group.

Empirically, the standard transition probability for sequential spin-½ measurements [1, 5] is

$$P(\beta) = \cos^2\left(\frac{\beta}{2}\right) = \frac{1}{2}(1 + cos\beta). \tag{3.2}$$

Because this angular dependence contains only two harmonic components (a constant term and a term linear in $cos\beta$, the prepared density $\rho(\mu)$ cannot contain any Legendre components beyond $l = 1$. As demonstrated rigorously in Appendix A, normalization and operational transition rules uniquely fix this marginalized post-selection density as:

$$\rho(\mu) = \frac{1}{4\pi} + \frac{1}{2\pi}\cos\theta\,. \tag{3.3}$$

This density necessarily becomes negative for $cos\ \theta < -1/2$. This signed character is unavoidable: enforcing pointwise non-negativity would require the inclusion of higher-order harmonic components, which are mathematically excluded by the symmetry constraints.

Proposition (Uniqueness of the Stern–Gerlach Post-Selection Distribution)

Let $\rho(\hat{s})$be an axisymmetric directional amplitude density on S^2describing the ensemble prepared by post-selection in a Stern–Gerlach apparatus aligned with axis $\hat{z}$. Suppose that

1. $\rho$is rotationally invariant about $\hat{z}$,
2. the transition probability for a second Stern–Gerlach analyzer is obtained by integration over the corresponding hemisphere,
3. the observed transition law is

$$P(\beta) = \frac{1}{2}(1 + \cos\beta). \tag{3.4}$$

Then the directional amplitude distribution is uniquely given by

$$\rho(\hat{s}) = \frac{1}{4\pi} + \frac{1}{2\pi}(\hat{s}\cdot\hat{z}). \tag{3.5}$$

Proof:
Axisymmetry implies $\rho(\hat{s}) = \rho(\mu)$ with $\mu = \cos\ \theta$, admitting a Legendre expansion

$$\rho(\mu) = \sum_{\ell=0}^{\infty} a_\ell\, P_\ell(\mu). \tag{3.6}$$

Because hemisphere integration is a rotation-covariant linear functional, each harmonic component contributes independently to the transition probability. The observed angular dependence contains only $P_0$and P_1, which excludes all $\ell \geq 2$ terms. Normalization and the requirement $P(0) = 1$ uniquely determine the coefficients. A detailed derivation is provided in Appendix A

4. Planar Marginalization and Photon Polarization

Photon polarization provides a two-dimensional analogue of the same geometric marginalization. A linear polarization state is represented by an axis in the transverse $x-y$ plane, specified by an angle φ, with the axial identification $\varphi \equiv \varphi + \pi$.

At a polarizer oriented at angle α, the action is a linear projection onto the transmission axis. The transmission amplitude is given by the scalar overlap $A(\alpha|\varphi) = \cos(\alpha - \varphi)$.

Because the configuration space is one-dimensional and axial, any admissible amplitude must be a periodic function of $\alpha - \varphi$ with period π. The lowest nontrivial harmonic consistent with this symmetry is $\cos(\alpha - \varphi)$. Higher harmonics would correspond to additional angular structure not supported by the operational action of a linear polarizer. Retaining only the lowest harmonic yields the unique minimal amplitude consistent with symmetry and geometric marginalization. Squaring this amplitude produces Malus' law [7] ($P = \cos^2(\alpha - \varphi)$) consistent with the Born rule [8].

5. Bipartite Amplitude Structure and Harmonic Reduction

We now extend this geometric marginalization to bipartite systems, demonstrating how global rotational invariance uniquely restricts the joint amplitude structure responsible for saturating the Tsirelson bound [3].

Consider a pair of spin-½ particles emitted from a common source. Let $\hat{s}_1, \hat{s}_2 \in S^2$ denote the directional amplitude parameters. The joint amplitude structure is represented by a normalized function $\rho(\hat{s}_1, \hat{s}_2)$ on the product space $S^2 \times S^2$.

For a phase-coherent, singlet-type preparation, no spatial direction is preferred. We therefore impose global rotational invariance: $\rho(R\hat{s}_1, R\hat{s}_2) = \rho(\hat{s}_1, \hat{s}_2$ for all $R \in SO(3)$. Under simultaneous rotations, the only nontrivial scalar invariant constructed from $\hat{s}_1$ and $\hat{s}_2$ is their scalar product, $\hat{s}_1 \cdot \hat{s}_2$. Consequently, rotational invariance marginalizes all other dependencies, implying $\rho(\hat{s}_1, \hat{s}_2) = \rho(\mathrm{u})$.

Observable joint statistics arise from bilinear pairing between this prepared structure and local measurement responses. Empirically, the correlation for a spin-singlet state is $E(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}$. Because both the preparation and the local measurement rules are rotationally covariant, the bilinear pairing between the joint amplitude distribution and the measurement responses preserves the harmonic decomposition of the distribution. The experimentally observed singlet correlation

$$E(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b} \tag{5.1}$$

contains only the first harmonic $P_1(\hat{a} \cdot \hat{b})$. Consequently, the joint amplitude distribution can contain no Legendre components beyond $\ell = 1$. The explicit determination of the coefficients is given in Appendix B. This mathematically forces the unique joint amplitude structure to take the signed form:

$$\rho(\hat{s}_1, \hat{s}_2) = \frac{1}{16\pi^2} - \frac{1}{4\pi^2}(\hat{s}_1 \cdot \hat{s}_2). \tag{5.2}$$

Substituting this uniquely determined amplitude structure into the bilinear pairing with local Stern–Gerlach response functions reproduces the correlation $E(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}$, which in turn yields the maximal quantum violation of the CHSH inequality. The earlier work therefore appears as a direct consequence of the symmetry and geometric marginalization principles established here.

6. Conclusions

In this work, we have shown that quantum state reduction and the specific signed amplitude distributions required to recover bipartite quantum correlations are not ad-hoc modeling choices. They are the rigorous mathematical consequences of geometric marginalization.

When a particle interacts with an apparatus, degrees of freedom orthogonal to the interaction plane $\boldsymbol{\Pi}(\hat{s}\ ,\hat{a})$ are operationally inaccessible and marginalized out of the effective description. We have proven via harmonic analysis that this marginalization—subject to rotational covariance and normalization—strictly forbids higher-order angular harmonics. This mathematically forces the emergence of the specific continuous, signed amplitude distributions utilized in our previous benchmarking of the Tsirelson bound.

By identifying orientation as the primitive kinematic variable and recognizing the unique role of measurement-defined marginalization, "collapse" is understood not as an exceptional dynamical process acting on reality, but as a restriction of description imposed by the geometry of interaction. This provides a coherent, kinematically complete, and mathematically unique account of measurement update that clarifies the structural origins of quantum statistics.

Appendix A: Symmetry and Uniqueness of the Stern-Gerlach Post-Selection Distribution

This appendix establishes the uniqueness of the directional amplitude distribution associated with post-selection in a Stern-Gerlach experiment [6], showing that no alternative angular dependence is compatible with the imposed symmetry and operational constraints.

A.1 Axisymmetric amplitude distributions on the sphere

Let $\hat{s}$ denote a unit vector on the sphere S^2, representing the directional spin amplitude of an individual particle. After post-selection by a Stern-Gerlach analyzer oriented along a fixed axis $\hat{z}$, the prepared ensemble is characterized by a real, normalized directional amplitude density $\rho(\hat{s})$ on S^2.

Because the preparation singles out the axis $\hat{z}$ but is otherwise invariant under rotations about that axis, ρ may depend only on the polar angle θ between $\hat{s}$ and $\hat{z}$, or equivalently on $\mu = cos\ \theta$. No dependence on the azimuthal angle is permitted. Thus,

$$\rho(\hat{s}_1, \hat{s}_2) \equiv \rho(u). \quad (A.1)$$

Any square-integrable axisymmetric function on the sphere admits a Legendre expansion,

$$\rho(u) = \sum_{\ell=0}^{\infty} a_\ell \, P_\ell(u), \quad (A.2)$$

where P_l are the Legendre polynomials and the coefficients a_l are real. Normalization over the sphere imposes the condition

$$\int_{S^2} \rho(\hat{s}) d\Omega = 2\pi \int_{-1}^{1} \rho(\mu) \, d\mu = 1, \quad (A.3)$$

which immediately fixes the coefficient of the $l = 0$ term once all others are specified.

A.2 Operational definition of transition probabilities

Consider a second Stern-Gerlach analyzer oriented along a unit vector $\hat{n}$ making an angle β with $\hat{z}$. Operationally, the analyzer transmits those components whose directional amplitudes satisfy $\hat{s} \cdot \hat{n} \geq 0$, corresponding geometrically to a hemisphere $H(\hat{n}) \in S^2$. The transition probability is defined as the integral of the prepared density over this hemisphere:

$$P(\beta) = \int_{H(\hat{n})} \rho(\hat{s}) d\Omega. \quad (A.4)$$

A.3 Harmonic content and constraints

Because the hemisphere integral is a rotation-covariant linear functional, it cannot mix distinct irreducible representations of the rotation group. Each Legendre component $P_l(\mu)$ in ρ therefore contributes independently to the angular dependence of $P(\beta)$:

$$\int_{H(\hat{n})} P_l(\hat{s} \cdot \hat{z}) d\Omega = C_l P_l(\hat{n} \cdot \hat{z}) = C_l P_l(\cos \beta), \quad (A.5)$$

where C_l is a numerical coefficient depending only on l.

Empirically, the transition probability for sequential Stern-Gerlach measurements on a spin-½ system is

$$P(\beta) = \cos^2\left(\frac{\beta}{2}\right) = \frac{1}{2}(1 + cos\beta). \quad (A.6)$$

This angular dependence contains only a constant term ($l = 0$) and a term proportional to $cos\ \beta = P_1(cos\ \beta)$.

This implies that the prepared density $\rho(\mu)$ cannot contain any Legendre components beyond $l = 1$. If any coefficient a_l with $l \geq 2$ were nonzero, the corresponding contribution would necessarily appear in $P(\beta)$, contradicting the observed form for arbitrary analyzer orientations. Thus, the most general admissible density is:

$$\rho(\mu) = a_0 + a_1\mu \quad (A.7)$$

A.4 Fixing the coefficients

Normalization fixes the constant term. Using

$$\int_{-1}^{1} \rho(\mu)\, d\mu = \frac{1}{2\pi}, \quad (A.8)$$

we find:

$$1 = 2\pi \int_{-1}^{1} a_0\, d\mu = 4\pi a_0 \;\to\; a_0 = \frac{1}{4\pi}, \quad (A.9)$$

The remaining coefficient is fixed by the requirement that a second analyzer aligned with the preparation axis transmits with unit probability, i.e., $P(0) = 1$. For $\beta = 0$, the transmitted region is the hemisphere $\mu \geq 0$, and

$$P(0) = 2\pi \int_{0}^{1} (\frac{1}{4\pi} + a_1 \mu) d\mu = \frac{1}{2} + \pi a_1, \quad (A.10)$$

Setting this equal to unity yields $a_1 = \frac{1}{2\pi}$. The unique post-selection density is therefore:

$$\rho(\mu) = \frac{1}{4\pi} + \frac{1}{2\pi}\cos\theta \quad (A.11)$$

Appendix B: Harmonic Structure of Bipartite Amplitudes on $S^2 \times S^2$

This appendix establishes the general form and uniqueness of the joint amplitude structure used to describe bipartite spin-½ systems.

B.1 Configuration space and symmetry assumptions

Consider a pair of spin-½ particles emitted from a common source. Let $\hat{s}_1, \hat{s}_2 \in S^2$ denote the directional amplitude parameters. The joint amplitude structure is represented by a normalized function $\rho(\hat{s}_1, \hat{s}_2)$ on the product space $S^2 \times S^2$, satisfying:

$$\int_{S^2} \int_{S^2} \rho(\hat{s}_1, \hat{s}_2)\, d\Omega_1 d\Omega_2 = 1. \quad (B.1)$$

For an entangled singlet-type preparation, no spatial direction is preferred. We therefore impose global rotational invariance:

$$\rho(R\hat{s}_1, R\hat{s}_2) = \rho(\hat{s}_1, \hat{s}_2) \quad \forall R \in SO(3). \quad (B.2)$$

B.2 Reduction to a function of the scalar product

Under simultaneous rotations of both arguments, the only nontrivial scalar invariant constructed from $\hat{s}_1$ and $\hat{s}_2$ is their scalar product, $u = \hat{s}_1 \cdot \hat{s}_2 \in [-1,1]$. Consequently, rotational invariance implies:

$$\rho(\hat{s}_1, \hat{s}_2) \equiv \rho(\mu). \tag{B.3}$$

Thus the joint amplitude structure on $S^2 \times S^2$ reduces to a single real function defined on the interval $[-1, 1]$.

B.3 Harmonic expansion and constraints

Any sufficiently regular function of u admits an expansion in Legendre polynomials:

$$\rho(\mu) = \sum_{\ell=0}^{\infty} a_\ell\, P_\ell(\mu). \tag{B.4}$$

Normalization fixes the zeroth-order coefficient:

$$1 = 16\pi^2 a_0 \quad \rightarrow \quad a_0 = \frac{1}{16\pi^2}. \tag{B.5}$$

Let $\hat{a}$ and $\hat{b}$ denote the orientations of two Stern-Gerlach analyzers acting locally on particles 1 and 2. Because both the preparation and the measurement rules are rotationally covariant, a term proportional to $P_l(u)$ generates a contribution proportional to $P_l(\hat{a} \cdot \hat{b})$ in the correlation function.

Empirically, the correlation for a spin-singlet state is $E(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}$. This dependence contains only the first-order harmonic $P_1(\hat{a} \cdot \hat{b})$. This immediately implies that all coefficients a_l with $l \geq 2$ must vanish. Thus the joint amplitude structure is restricted to:

$$\rho(\mu) = a_0 + a_1 u. \tag{B.6}$$

B.4 Fixing the correlation coefficient

The coefficient a_1 is fixed by matching the magnitude of the observed correlation. Substituting the truncated form into the bilinear pairing defining the correlation function yields

$$E(\hat{a}, \hat{b}) = K a_1 (\hat{a} \cdot \hat{b}), \tag{B.7}$$

where K is a positive numerical constant determined by the geometry of the hemisphere selection. Evaluating this constant explicitly yields $K = 4\pi^2$.

Matching the singlet correlation then gives $4\pi^2 a_1 = -1$, so $a_1 = -1/4\pi^2$. The unique joint amplitude structure consistent with rotational invariance and the observed correlation is therefore:

$$\rho(\hat{s}_1, \hat{s}_2) = \frac{1}{16\pi^2} - \frac{1}{4\pi^2} (\hat{s}_1 \cdot \hat{s}_2). \tag{B.8}$$

This signed structure encodes phase-coherent relational information established at preparation and gives rise to the observed correlations under strictly local measurements.

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